

A Possible Generalization of Quantum Mechanics

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A “minimal” generalization of Quantum Mechanics is proposed, where the Lagrangian or the action functional is a mapping from the (classical) states of a system to the Lie algebra of a general compact Lie group, and the wave function takes values in the corresponding group algebra. This formalism admits a probability interpretation and a suitable dynamics, but has no obvious classical correspondence. Allowing the Lagrangian or the action functional to take values in a general Lie algebra instead of only the real number field (actually the $u(1)$ algebra) enlarges the extent of possible physical laws that can describe the real world. The generalized quantum dynamics of a point particle in a background gauge field is given as an example, which realizes the gauge invariance by a Wilson line structure and shows that some Schrödinger-like equation can be deduced within this formalism. Some possible developments of this formalism are also discussed.

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Quantum Mechanics (QM) has become one of the most important foundations of physics ever since its establishment in the 1920s. While QM was achieving great success, physicists had begun to consider the possibility to generalize the traditional framework of it [1]. Up to now, although various generalizations of QM are all not very useful and/or successful, this kind of attempts have never stopped (see, for example, Refs. [2, 3, 4, 5] and references therein). This letter proposes a new way to generalize QM, which makes few changes in the formalism of QM and may be viewed as a minimal generalization of QM. The greatest change caused by this generalization may be that it renders QM to have no obvious classical correspondence. However, this may not be a shortcoming of the generalized QM, but rather an advantage in some sense. In fact, although in the present framework of quantum theory (including also quantum field theory and even string theory) it is taken for granted that there should first be a classical system and then it is quantized to give real physics, it has long been suspected that our universe may eventually be governed by some intrinsic quantum structure, i.e., there may be no corresponding classical structure for the ultimate law of nature at all. The generalized QM here may play such a role. As a simple example, the generalized quantum dynamics of a point particle in a background gauge field is proposed, which has some advantage over the ordinary treatments. This example shows that the generalized fundamental principles of QM can lead to some Schrödinger-like equation with a formal Hamiltonian, but it is not the ordinary quantization of a Hamilton’s mechanics because the generalized quantum system has no real-valued Lagrangian. Interestingly, the gauge invariance in this model is naturally realized by a Wilson line structure of the interaction.

In order to see why and how we can generalize QM, first let us briefly review the fundamental principles of (non-relativistic) QM. There are many versions of these principles. Here we only list one from the viewpoint of the Schrödinger picture:[10]

1. The quantum state of a system is described by the wave function ψ , [11] which is a complex function of time t and the canonical coordinates (collectively denoted by \mathbf{r}) of the system; at time t , the probability to find the system in the classical configuration \mathbf{r} is proportional to $|\psi(t, \mathbf{r})|^2 d\mathbf{r}$.
2. A mechanical quantity (observable) F is a Hermitian operator acting on the Hilbert space formed by the set of wave functions, which has a complete set of eigenfunctions; the only possible result of one measurement of F is one of its eigenvalues $\{f_n\}$.
3. If the wave function of the system is the n -th eigenfunction ψ_n of F , then the result of any measurement of F is the corresponding eigenvalue f_n ; if the wave function is a linear superposition $\sum_m a_m \psi_m$, then the probability to get a measuring result f_n is $|a_n|^2 / \sum_m |a_m|^2$.
4. The dynamics of the wave function is governed by the Schrödinger equation, which is a linear differential equation.
5. The hypothesis of wave-packet collapse after measurements, whose details are not relevant here, or other measurement theories for QM.

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These principles are easy to understand and sufficient to establish the whole framework of QM. A useful alternative of Principle 4 is the so-called Feynman's hypothesis of path integral. It is well known that there are difficulties to set up the Feynman path integral on firm mathematical foundation. However, a version of the Feynman path integral in terms of infinitesimal time evolution seems less troublesome. The dynamics of the wave function is described as a Huygens-like principle, which determines the wave function at time $t + \epsilon$ from the wave function at time t as

$$\psi(t + \epsilon, \mathbf{r}) = \int d\mathbf{r}' K(t + \epsilon, \mathbf{r}; t, \mathbf{r}') \psi(t, \mathbf{r}') \quad (1)$$

with $K(t + \epsilon, \mathbf{r}; t, \mathbf{r}')$ the propagator. Unlike the Feynman's general hypothesis that assumes the propagator to be a complicated path integral, for infinitesimal ϵ the propagator is assumed to be ($\hbar = 1$)

$$K(t + \epsilon, \mathbf{r}; t, \mathbf{r}') = N \exp \left[i\epsilon L \left(\frac{\mathbf{r} + \mathbf{r}'}{2}, \frac{\mathbf{r} - \mathbf{r}'}{\epsilon}, t \right) \right] \quad (2)$$

with $L(\mathbf{r}, \dot{\mathbf{r}}, t)$ the Lagrangian and N an appropriate normalization factor. This infinitesimal version of Feynman path integral is actually equivalent to the so-called "polygonal paths" approach proposed by Feynman, and can be shown to be equivalent to the Schrödinger equation.

As shown by the above fundamental principles, there are two kinds of quantities that play important roles in QM. One is the wave function, called also the probability amplitude, which is viewed as the most fundamental; the other is the probability, which is less fundamental but directly related to observations (measurements). The probability interpretation (Principle 1 and 3) of QM just states that there is a mapping from the former to the latter. Since the probability must be a real number, there is no obvious way to generalize the notion of probability. However, the notion of probability amplitude is much more flexible, provided there exists a mapping from it to the probability. That is the key point to make our minimal generalization which keeps the other parts of QM almost intact. In fact, many previous attempts to generalize QM just follow this path [2, 3].

Now the observation is that we can view the (usual) probability amplitude as taking values in the $U(1)$ group algebra, instead of the complex number field as usual. These two viewpoints are equivalent so far, but the one of group algebra accepts much further generalization. Obviously, the next step is to allow the probability amplitude to take values in a general Lie group algebra, where the group is restricted to be compact for simplicity in this letter. Moreover, we must define the mapping from this group algebra to the probability so that the generalized theory can adopt a probability interpretation. This can be easily done with the aid of group representation theory. Given that any compact Lie group G has finite dimensional unitary representations, and assuming the mapping to be quadratic, we can define the mapping from any $g \in \mathcal{G}$ (the corresponding group algebra of G) to the probability p as

$$p(g) = \text{Tr}(g^\dagger g). \quad (3)$$

Here (and in the following) we have used the same notation for g and its representation, and the trace is taken in some fixed representation. Compatible with the above mapping, an inner product on the group algebra space can be immediately defined:

$$(g_1, g_2) = \text{Tr}(g_1^\dagger g_2), \quad (4)$$

so that $p(g)$ can be identified with (g, g) . Alternatively, we can use real representations (by orthonormal matrices) of G and define

$$(g_1, g_2) = \text{Tr}(g_1^T g_2), \quad (5)$$

where the superscript T stands for matrix transposition.

In order to establish a complete set of fundamental principles as before, we further assume that a mechanical quantity F is still a Hermitian operator on the ordinary Hilbert space, but not an operator on the group algebra space. The time operator t and the coordinate operator \mathbf{r} , which have continuous spectrums of eigenvalues, are also included in this statement. The whole wave function space is thus a direct product of the ordinary Hilbert space and the group algebra space. The original inner product on the Hilbert space still works, which in general produces from the whole wave function space a result on the group algebra space.

So far, our generalization is only notional. But it is indeed possible for this generalization to have corresponding dynamics. If we would like Eq. (1) to govern as usual the dynamics of the generalized QM, where now the wave function $\psi(t, \mathbf{r})$ takes values in \mathcal{G} , it is obvious that the propagator K can also take values in \mathcal{G} . If, further, the infinitesimal propagator is still assumed to be given by Eq. (2), the Lagrangian should take values in the Lie algebra \mathfrak{g} of G . In other words, the generalized Lagrangian is a mapping

$$L : (\mathbf{r}, \dot{\mathbf{r}}, t) \rightarrow \mathfrak{g}. \quad (6)$$

Correspondingly, in the Feynman's general hypothesis of path integral, the action functional should take values in \mathfrak{g} . In fact, infinitely many successive propagators (2) can be sewn together to express the finite time propagator as a generalized path integral

$$K(t_2, \mathbf{r}_2; t_1, \mathbf{r}_1) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathcal{T} \left[\exp \int_{t_1}^{t_2} L(\mathbf{r}, \dot{\mathbf{r}}, t) dt \right] \mathcal{D}\mathbf{r}(t) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathcal{T}(e^{S[\mathbf{r}]}) \mathcal{D}\mathbf{r}(t), \quad (7)$$

where \mathcal{T} denotes time ordering and the normalization factor has been absorbed into the definition of $\mathcal{D}\mathbf{r}(t)$. The above equation also shows that the naive generalization of the Feynman path integral is incorrect unless modified by the time ordering procedure. The Lagrangian (action functional) in the ordinary QM takes values in \mathbb{R} , which is just the Lie algebra of the U(1) group. In this case the time ordering operator acts trivially, so the generalized Feynman path integral becomes the ordinary one.

To sum up, our fundamental principles of the generalized QM, in comparison with that of the ordinary QM stated above, are

1. The same as before, but now $|\psi(t, \mathbf{r})|^2 d\mathbf{r}$ is replaced with $p[\psi(t, \mathbf{r})] d\mathbf{r}$.
2. The same as before.
3. The same as before, but now a_n takes values in the group algebra and $|a_n|^2$ is replaced with $p(a_n)$.
4. The dynamics of the wave function is governed by Eqs. (1) and (2), where ψ , K and L all have the generalized meaning.
5. The same as before.

A nice feature of our generalization is that a generalized quantum system can be consistently coupled to an ordinary one simply because the latter is a special case of the former. In fact, this can be viewed as an important requirement for the generalization to be physically acceptable, since most of the known physical phenomena can be perfectly described by the ordinary quantum theory. With the above feature, our generalized quantum theory may describe some unclear physics or act as tiny corrections to known physics. An interesting case is the U(N)-generalized Lagrangian

$$L = iL_0 + L_{\text{SU}(N)}, \quad (8)$$

where L_0 is the ordinary real-valued Lagrangian and the SU(N)-valued part $L_{\text{SU}(N)}$ acts as some perturbation. Anyway, extending the range of Lagrangian or action functional to a general Lie algebra enables one to have more choices of possible physical laws for describing the real world. One may wonder how we can have a natural mapping from a physical system to a general Lie algebra. The answer is that physical systems themselves have Lie algebras as symmetry structures.

Let us consider a point particle in a background gauge field. This system was first considered by Wong [7], and then discussed further by many other authors (see Ref. [8] and references therein and thereafter). The generalized Lagrangian can be taken as

$$L = \frac{i}{2}(v^{-1}\dot{x}^2 - vm^2) - A_\mu \dot{x}^\mu, \quad \dot{x}^2 = \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad (9)$$

for relativistic case, where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, v is the Lagrange multiplier enforcing the mass-shell condition and A_μ the anti-Hermitian gauge potential. The gauge group is arbitrary, though it can be assumed as SU(N) in order for comparison with Eq. (8). Since the interaction term in Eq. (9) commutes with the free part of the Lagrangian, substitution of Eq. (9) into Eq. (7) shows that the interaction term just produces a Wilson line factor in the path integral:

$$\int_{x_1}^{x_2} \mathcal{T} \left(\exp \int_{t_1}^{t_2} L dt \right) \mathcal{D}x(t) = \int_{x_1}^{x_2} e^{iS_0} W[x; A] \mathcal{D}x(t), \quad W[x; A] = \exp \left(- \int_{t_1}^{t_2} A_\mu dx^\mu \right), \quad (10)$$

where S_0 is the (real-valued) free action functional. Under gauge transformation

$$\tilde{A}_\mu = U A_\mu U^{-1} + U \partial_\mu U^{-1}, \quad (11)$$

the Wilson line transforms as (see, for example, Ref. [9])

$$W[x; \tilde{A}] = U W[x; A] U^{-1}. \quad (12)$$

Thus the propagator transforms similarly, which leads to the gauge invariance of any physical results of this model. This realization of the gauge invariance seems much easier than those before, where additional degrees of freedom are inevitably introduced [8].

Further, to make our generalization physically interesting and to show that the generalized Principle 4 can really lead to some Schrödinger-like equation as before, we use the above model as an example. For simplicity in the following computation, we take the non-relativistic limit of the Lagrangian (9) and restrict the spatial dimension to be one:

$$L(x, \dot{x}, t) = \frac{i}{2} m \dot{x}^2 - A \dot{x} - \varphi, \quad (13)$$

where we have defined $\varphi = A_0$ and $A = A_1$. Now we can apply a standard approach (see, for example, Ref. [6]) to convert Eq. (1) into the form of differential equation. According to Eq. (2) we have

$$K(t + \epsilon, x; t, x') = N \exp \left[im \frac{(x - x')^2}{2\epsilon} - (x - x')A \left(\frac{x + x'}{2} \right) - \epsilon \varphi \left(\frac{x + x'}{2} \right) \right] \quad (14)$$

$$\approx N \exp \left[\frac{im}{2\epsilon} \left(x' - x - \frac{i\epsilon}{m} A \right)^2 \right] \left(1 + \frac{i\epsilon}{2m} A^2 - \epsilon \varphi \right). \quad (15)$$

From Eq. (14) to Eq. (15), the Baker-Campbell-Hausdorff formula should be applied, but the extra terms are all infinitesimals of higher order and so can be neglected. Substitution of Eq. (15) into Eq. (1) gives

$$\psi(t, x) + \epsilon \partial_t \psi(t, x) = N \int d\xi \exp \left[\frac{im}{2\epsilon} \left(\xi - \frac{i\epsilon}{m} A \right)^2 \right] \left(1 + \frac{i\epsilon}{2m} A^2 - \epsilon \varphi \right) \left[\psi(t, x) + \xi \partial_x \psi(t, x) + \frac{\xi^2}{2} \partial_x^2 \psi(t, x) \right], \quad (16)$$

where we have defined

$$\xi = x' - x \quad (17)$$

and neglected all higher order terms in the Taylor expansions. The zeroth order of Eq. (16) reads

$$N \int \exp \left(\frac{im}{2\epsilon} \xi^2 \right) d\xi = 1, \quad (18)$$

so it follows that

$$N \int \exp \left\{ \frac{im}{2\epsilon} \left[\xi - \frac{i\epsilon}{m} A \left(x + \frac{\xi}{2} \right) \right]^2 \right\} d\xi \approx N \int \exp \left\{ \frac{im}{2\epsilon} \left[\left(1 - \frac{i\epsilon}{2m} \partial_x A \right) \xi - \frac{i\epsilon}{m} A \right]^2 \right\} d\xi \quad (19)$$

$$\approx 1 + \frac{i\epsilon}{2m} \partial_x A. \quad (20)$$

Knowing also

$$N \int \xi \exp \left[\frac{im}{2\epsilon} \left(\xi - \frac{i\epsilon}{m} A \right)^2 \right] d\xi = N \int \frac{i\epsilon}{m} A \exp \left[\frac{im}{2\epsilon} \left(\xi - \frac{i\epsilon}{m} A \right)^2 \right] d\xi \approx \frac{i\epsilon}{m} A \quad (21)$$

and

$$N \int \xi^2 \exp \left[\frac{im}{2\epsilon} \left(\xi - \frac{i\epsilon}{m} A \right)^2 \right] d\xi = N \int \left[\left(\xi - \frac{i\epsilon}{m} A \right)^2 - \frac{\epsilon^2}{m^2} A^2 \right] \exp \left[\frac{im}{2\epsilon} \left(\xi - \frac{i\epsilon}{m} A \right)^2 \right] d\xi \quad (22)$$

$$\approx N \int \frac{i\epsilon}{m} \exp \left[\frac{im}{2\epsilon} \left(\xi - \frac{i\epsilon}{m} A \right)^2 \right] d\xi \approx \frac{i\epsilon}{m} \quad (23)$$

from integration by parts, Eq. (16) then becomes

$$\hbar \partial_t \psi(t, x) = \left[\frac{i\hbar^2}{2m} \partial_x^2 + \frac{i\hbar}{m} A \partial_x + \frac{i\hbar}{2m} (\partial_x A) + \frac{i}{2m} A^2 - \varphi \right] \psi(t, x) = \left[\frac{i}{2m} (\hbar \partial_x + A)^2 - \varphi \right] \psi(t, x), \quad (24)$$

where we have restored the Planck constant omitted in Eq. (2). This Schrödinger-like equation is gauge invariant if the wave function transforms as

$$\tilde{\psi}(t, x) = U\psi(t, x). \quad (25)$$

Thus we have easily obtained a satisfactory quantum dynamics of this system, while the ordinary quantization procedure is much more complicated [8].

Nevertheless, the Schrödinger-like equation (24) can be viewed as a standard Schrödinger equation with a multi-component $\psi(t, x)$ and a Hamiltonian

$$\mathcal{H} = -\frac{1}{2m}(\hbar\partial_x + A)^2 - i\varphi. \quad (26)$$

But the point is that the generalized quantum system does not have a real-valued Lagrangian, so its classical correspondence is not clear [12] and the above Hamiltonian is just formal if it has nothing to do with the (generalized) Lagrangian. In this example, we may formally define the canonical momentum conjugate to x as

$$p = \frac{\partial L}{\partial \dot{x}} = im\dot{x} - A, \quad (27)$$

which takes values in the Lie algebra. Thus the “classical Hamiltonian” is

$$H = \dot{x}p - L = \frac{i}{2}m\dot{x}^2 + \varphi = -\frac{i}{2m}(p + A)^2 + \varphi, \quad (28)$$

which is proportional to the “quantum Hamiltonian” (26) if we “quantize” p as $\hbar\partial_x$. However, it is not yet obvious how to understand a Lie-algebra-valued canonical momentum in classical mechanics.

Several aspects of this formalism can be studied further. The Lagrangian (13) of our model is simple enough, so more general cases should be investigated, which is not easy. Applications of more physical meaning are still lacking. The possibility of existence of some classical dynamics corresponding to the generalized quantum system can be carefully examined. Moreover, natural developments of this formalism, including quantum field theory, quantum statistical mechanics, quantum information and so on, are worthy of future work.

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 - [10] Neither the strictness in mathematics nor the axiomatization of physics is pursued here. These principles are listed only for the purpose of generalization.
 - [11] Here we do not use the notion “ray”. In fact, the usual definition of ray should be modified under our generalization.
 - [12] In fact, it is easy to see that the ordinary Euler-Lagrange equation deduced from the Lagrangian (9) or (13) is inconsistent unless the gauge potential takes values in $\mathfrak{u}(1)$.